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Rotational effects in Stokes flow; pressure-driven extrusion through an annular hole or concentric holes in parallel walls

A.M.J. DAVIS
Mathematics Department, University of Alabama, Tuscaloosa, AL 35487-0350, U.S.A.
(e-mail: adavis@euler.math.ua.edu)

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Abstract. This tribute to the late Prof. L.G.N. Filon, who preceded the author as a faculty member of University College London, is devoted to illustrating the application of Abel transform techniques to mixed boundary problems. The two situations discussed below, the pressure-driven extrusion through an annular hole in a plane wall and concentric holes in two planes, have been previously studied as creeping flows but here they are complicated by being set in a viscous fluid undergoing rigid-body rotation. The analysis, adapted from calculations of the Stokes drag on a disk sedimenting in a rotating fluid in the presence of fixed boundaries, may not have direct engineering application but presents classic techniques applied to a set of triple integral equations and a system of dual integral equations.

Key words: extrusion, integral equations, mixed-boundary-value problems, rotating viscous fluid, Stokes flow

1. Introduction

Many years after the Taylor column was first observed [1], several authors [2–5] investigated the slow axial motion of a sphere or ellipsoid in a slightly viscous, rotating fluid. Maxworthy [6, 7] reported experimental results pertaining to these studies and also to the viscosity-dominated, small-Taylor-number case, considered theoretically by Childress [8]. Weisenborn [9] gave results that appear to be valid for all Taylor numbers T, the ratio of Coriolis to viscous forces. With the governing equations non-separable, a boundary-integral approach was used by Tanzosh and Stone [10] for the drag and velocity fields for axial translation of spheres and ellipsoids in rotating viscous fluids with $0 \leq T \leq 1000$. Noticing that disk-and-plane geometries allow the use of separated functions, Vedensky and Ungarish [11], who fully described the above-mentioned work, and Ungarish and Vedensky [12] considered the motion generated by a slowly rising disk in a rotating fluid that is either unbounded or axially bounded by rotating planes between which the disk is instantaneously centrally placed. With a similar motive and method, Tanzosh and Stone [13] investigated the corresponding motion generated when the disk moves in its own plane at right angles to the rotation axis of the unbounded fluid. These authors used Tranter's method [14, Section 4.6.3] to solve their dual integral equations and chose the order of the edge singularity, which was otherwise obvious, to achieve the best convergence with respect to the truncation of the system of linear equations. Here it is important, of course, to convert from equations of the first kind to those of the second kind. Although Tranter's method uses an infinite sum involving Bessel functions whose arguments are proportional to $T^{1/2}$, an efficient scheme for evaluating integrals of products of such Bessel functions furnished good agreement with both small- and large-Taylor-number, T, results and allowed these authors to give their main attention to the physically more interesting case of large T. However, the Tranter method cannot cope with a concentric cylindrical bounding wall for which, along with the above disk-and-plane combinations, Davis [15] reduced sets of integral equations to sets of integral equations of the second kind by using, as in earlier work [16–18], the Abel transforms of the pressure, vorticity and tangential-stress discontinuities at the disk and thereby fixing the order of the rim singularity in each of these jumps. The method mimicks the boundary-integral method by being equivalent to using distributions, on the disk, of force singularities which can be readily modified to take account of rigid boundaries. However, the convergence of the numerical solution of the integral equations becomes slower as T increases and so only values of T up to 50 could be considered. For larger T, the Tranter method was used, where applicable, by Davis and Stone [19] who also considered disk oscillations.

Here the interaction of rotation with pressure-driven flow through a hole in one or more planes is investigated. The single plane flow, which is reworked to furnish a simple illustration of the Tranter method, is used as the basis for the annular hole and two plane solutions, as in [20] and [17], respectively. Otherwise, this analysis mimicks that of Davis [15], with the two-plane geometry being complementary to that of two disks. The flow through an annular hole is symmetric about the plane, thus having only a pressure jump which is handled, in terms of two Abel transforms, as in the non-rotating case [20]. The flow through two rotating planes requires Abel transforms for the vorticity and tangential-stress discontinuities, while the pressure jump at each plane is handled as for the single plane, a variation of [17].

2. Pressure-driven flow through a hole in a rotating wall

Incompressible viscous fluid that is rotating with the same angular velocity Ω as a rigid plane wall with a hole of radius R, is caused to flow through the hole by the imposition of a pressure drop ΔP. Dimensionless cylindrical polar coordinates (r, θ, z) are chosen to be rotating with the fluid at infinity and so that the hole is at $z = 0\,(a \leq r \leq 1, -\pi < \theta \leq \pi)$ and the flow is everywhere positive in the direction of the unit vector $\hat{\mathbf{z}}$. The Reynolds number is assumed to be sufficiently small for the velocity field $R\Delta P\mathbf{v}/\mu$ to satisfy the equations of almost rigid rotation [21, Chapter 1], which might be regarded as the Stokes equations referred to rotating axes,

$$2T\hat{\mathbf{z}} \times \mathbf{v} = -\nabla p + \nabla^2\mathbf{v}, \qquad \nabla \cdot \mathbf{v} = 0, \tag{1}$$

where $p\Delta P$ is the dynamic pressure, μ the coefficient of viscosity and, with ρ the density, the Taylor number T is defined by

$$T = \frac{\Omega R^2 \rho}{\mu} = \frac{Q^2}{2}. \tag{2}$$

The magnitude of T is irrelevant to the creeping relative flow approximation when there is no azimuthal dependence [4], since then the rigid-body rotation of the fluid cannot generate azimuthal convection. In the axisymmetric flows with swirl to be considered, two components of $\mathbf{v}$ can be expressed in terms of a stream function $\psi(r, z)$ by writing

$$v_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad v_z = \frac{1}{r}\frac{\partial \psi}{\partial r}. \tag{3}$$

Substitution of (3) in (1) then shows that the momentum equation requires the function $\psi(r, z)$ to be such that

$$\left[\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right]^3 \psi = L_{-1}^3 \psi = -4T^2 \frac{\partial^2 \psi}{\partial z^2}, \tag{4}$$

with the dimensionless pressure and swirl velocity successively determined according to

$$\frac{\partial p}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r} L_{-1}\psi, \tag{5}$$

$$2T v_\theta = \frac{\partial p}{\partial r} + \frac{1}{r}\frac{\partial}{\partial z} L_{-1}\psi. \tag{6}$$

The L_{-1} notation is taken from generalized axisymmetric potential theory and is often used for Stokes flows. The boundary conditions to be applied are

$$v_z = 0, \quad v_r = 0, \quad v_\theta = 0, \text{ at } z = 0 \quad (r > 1), \tag{7}$$

$$p \sim -\frac{1}{2}\operatorname{sgn} z \text{ as } |z| \to \infty, \tag{8}$$

$$[p]_{0-}^{0+} = 0, \text{ at } z = 0 \quad (0 \le r < 1), \tag{9}$$

corresponding, respectively, to no slip on the wall, the imposed pressure drop and continuity of pressure in the hole. The problem posed by the rotating system, with non-separable equations for ψ, v_θ but separate boundary conditions, differs from that posed by, for example, creeping transverse flow past a body of revolution in which functions ψ, ϕ are governed by $L_{-1}^2\psi = 0 = L_{-1}\phi$ but the boundary conditions are non-separable.

In the absence of rotation, the swirl velocity $v_\theta = 0$ in the Sampson flow [22] and (4) is replaced by $L_{-1}^2\psi = 0$, whose solution, for an everywhere continuous velocity field that is symmetric about $z = 0$, is given, in terms of the usual Hankel transform, by

$$\psi = \frac{r}{2\pi}\int_0^\infty A(k) J_1(kr)(k^{-1} + |z|)\mathrm{e}^{-k|z|}\,\mathrm{d}k. \tag{10}$$

Since the only boundary is at $z = 0$, the simpler representation,

$$\mathbf{v} = \phi\hat{\mathbf{z}} - z\nabla\phi, \qquad p = -2\frac{\partial\phi}{\partial z},$$

where $\nabla^2\phi = 0$, suffices, with

$$\phi = \frac{1}{4}\left[|z| + \frac{2}{\pi}\int_0^\infty A(k) J_0(kr)\mathrm{e}^{-k|z|}\,\mathrm{d}k\right]. \tag{11}$$

The two formulations yield the same conditions on $A(k)$, whence it is found that

$$A(k) = \frac{\mathrm{d}}{\mathrm{d}k}\left(-\frac{\sin k}{k}\right), \quad [p]_{0-}^{0+} = -\frac{2}{\pi}\left[\cos^{-1}\left(\frac{1}{r}\right) + \frac{1}{\sqrt{r^2-1}}\right] \quad (r > 1), \tag{12}$$

and the dimensionless flux is 1/3. Note the square-root rim singularity in the pressure jump.

The use of the Laplacian ϕ is precluded by rotation and the solution of (4) requires the prior definition, for $\kappa > 0$, of $\lambda_1(\kappa)$ $(-1 < \lambda_1 < 0)$, $\lambda_2(\kappa)$ and $\lambda_3(\kappa)$ $(\lambda_3 = \overline{\lambda_2})$ to be the roots of

$$\lambda^3 + \kappa^{-4}(\lambda + 1) = 0. \tag{13}$$

Then, on setting

$$\eta_1(\kappa) = \lambda_1(\lambda_2 - \lambda_3), \qquad \eta_2(\kappa) = \lambda_2(\lambda_3 - \lambda_1), \qquad \eta_3(\kappa) = \lambda_3(\lambda_1 - \lambda_2),$$

$$\Pi(\kappa) = (\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)(\lambda_1 - \lambda_2), \tag{14}$$

it follows that

$$\sum_{j=1}^{3} \frac{\eta_j}{\lambda_j} = 0 = \sum_{j=1}^{3} \eta_j \tag{15}$$

and also, from (14) and (15), that

$$\sum_{j=1}^{3} \frac{\eta_j \lambda_j}{\lambda_j + 1} = \sum_{j=1}^{3} \eta_j \lambda_j = -\Pi. \tag{16}$$

With these functions and identities, it may be shown that a solution of (4) that yields an everywhere continuous velocity field that is symmetric about $z = 0$, is given by

$$\psi = \frac{r}{\pi} \int_0^\infty \frac{J_1(kr)}{k\Pi(k/Q)} A(k) \sum_{j=1}^{3} \frac{\eta_j(k/Q)}{\sqrt{\lambda_j + 1}} \mathrm{e}^{-k|z|\sqrt{\lambda_j+1}} \, \mathrm{d}k, \tag{17}$$

where Q is defined by (2) and the expression (10) is recovered in the limit $Q \to 0$ since $\lambda_j(\kappa) \to 0$ $(j = 1, 2, 3)$ as $\kappa \to \infty$. Then v_r and v_z are readily deduced from (3), while (5) and (6) imply that

$$v_\theta = \frac{Q^2}{\pi} \mathrm{sgn} z \int_0^\infty \frac{J_1(kr)}{k^2 \Pi(k/Q)} A(k) \sum_{j=1}^{3} \frac{\eta_j}{\lambda_j} \mathrm{e}^{-k|z|\sqrt{\lambda_j+1}} \, \mathrm{d}k,$$

$$p = -\mathrm{sgn} z \left[\frac{1}{2} + \frac{1}{\pi} \int_0^\infty J_0(kr) \frac{kA(k)}{\Pi(k/Q)} \sum_{j=1}^{3} \frac{\eta_j \lambda_j}{\lambda_j + 1} \mathrm{e}^{-k|z|\sqrt{\lambda_j+1}} \, \mathrm{d}k \right]. \tag{18}$$

Only the pressure has a discontinuity at $z = 0$, given, after using (16), by

$$[p]_{0-}^{0+} = -1 + \frac{2}{\pi} \int_0^\infty k J_0(kr) A(k) \, \mathrm{d}k.$$

This is confined to the wall by imposing the condition,

$$\frac{2}{\pi} \int_0^\infty k J_0(kr) A(k) \mathrm{d}k = 1 \quad (0 \le r < 1), \tag{19}$$

which, by use of the identity [23, Section 6.554],

$$\int_0^u J_0(kr) \frac{r \mathrm{d}r}{(u^2 - r^2)^{1/2}} = \frac{\sin ku}{k}, \tag{20}$$

may be expressed as

$$\frac{2}{\pi} \int_0^\infty A(k) \sin ku \, \mathrm{d}k = u \qquad (0 \le u < 1). \tag{21}$$

Symmetry of the flow about $z = 0$ ensures that v_r, v_θ satisfy the no-slip conditions (7), whose remaining requirement is, according to (3) and (17),

$$v_z(r, 0) = \frac{1}{\pi}\int_0^\infty \frac{J_0(kr)}{\Pi(k/Q)} A(k) \sum_{j=1}^{3} \frac{\eta_j(k/Q)}{\sqrt{\lambda_j + 1}} \mathrm{d}k = 0 \quad (r > 1). \tag{22}$$

Just as (17) reduces to (10) in the limit $Q \to 0$, so condition (22) reduces to $(2\pi)^{-1}\int_0^\infty J_0(kr)A(k)\mathrm{d}k = 0 (r > 1)$, in accordance with the Sampson flow. With this limit in mind, define

$$\Delta(\kappa) = 1 - \frac{2}{\Pi(\kappa)} \sum_{j=1}^{3} \frac{\eta_j(\kappa)}{\sqrt{\lambda_j(\kappa) + 1}} \sim \begin{cases} 1 - 2\kappa^2 & \text{as } \kappa \to 0 \\ 5/64\kappa^4 & \text{as } \kappa \to \infty \end{cases} \tag{23}$$

Use of (13) with the identities (15) and (16) shows that

$$\sum_{j=1}^{3} \eta_j \lambda_j^2 = 0, \quad \sum_{j=1}^{3} \eta_j \lambda_j^3 = \sum_{j=1}^{3} \eta_j \lambda_j^4 = -\frac{1}{k^4} \sum_{j=1}^{3} \eta_j \lambda_j = \frac{\Pi}{k^4}$$

and hence $\Delta(\kappa) = O(\kappa^{-4})$ as $\kappa \to \infty$. It can also be noted from (13) that

$$\lambda_1 \sim -1 + \kappa^4, \qquad \lambda_2 \sim \kappa^{-2}\mathrm{i} + \frac{1}{2} \text{ as } \kappa \to 0, \tag{24}$$

so that λ_1 increases from -1 to 0 as κ goes from 0 to ∞, with

$$\lambda_2 = -\frac{1}{2}\lambda_1 + \mathrm{i}(\frac{3}{4}\lambda_1^2 + \kappa^{-4})^{1/2}, \quad \lambda_3 = \overline{\lambda_2}. \tag{25}$$

Thus $\sqrt{\lambda_j + 1} \to 1 (j = 1, 2, 3)$ as $\kappa \to \infty$ while

$$\sqrt{\lambda_1 + 1} \sim \kappa^2 \quad \sqrt{\lambda_2 + 1} \sim \kappa^{-1}\mathrm{e}^{\mathrm{i}\pi/4} \text{ as } \kappa \to 0.$$

The condition (22) is automatically satisfied by setting, as in [20],

$$A(k) = \frac{2}{\Pi(k/Q)} \sum_{j=1}^{3} \frac{\eta_j(k/Q)}{\sqrt{\lambda_j + 1}} = \int_0^1 t^{-1} F(t) \sin kt \, \mathrm{d}t, \tag{26}$$

whose substitution in the pressure-continuity condition (21) yields, in terms of the function $\Delta(\kappa)$ defined by (23), the integral equation

$$\frac{F(u)}{u} + \frac{2}{\pi}\int_0^1 \frac{F(t)}{t} \int_0^\infty \sin ku \, \sin kt \frac{\Delta(k/Q)}{1 - \Delta(k/Q)} \mathrm{d}k\mathrm{d}t = u \quad (0 \leq u < 1). \tag{27}$$

The velocity profile in the hole and the consequent dimensionless flux are given, respectively, by

$$v_z(r, 0) = \frac{1}{2\pi}\int_r^1 \frac{F(t)\mathrm{d}t}{t(t^2 - r^2)^{1/2}}, \quad M = \int_0^1 F(t)\mathrm{d}t. \tag{28}$$

In the limit $Q \to 0$, (27) and (28) yield $F(t) = t^2$, $v_z(r, 0) = (2\pi)^{-1}(1-r^2)^{1/2}$ and $M = 1/3$, consistent with the Sampson (non-rotating) flow.

The integral equation (27) possesses an iterative solution for small enough values of Q. Indeed, the kernel

$$\frac{2}{\pi}\int_0^\infty \sin\, ku\ \sin\, kt\frac{\Delta(k/Q)}{1-\Delta(k/Q)}\mathrm{d}k \sim \frac{2Q^3ut}{\pi}\int_0^\infty \frac{\kappa^2\Delta(\kappa)}{1-\Delta(\kappa)}\mathrm{d}\kappa$$

and hence, at this order, $F(t)$ and thus $v_z(r,0)$ and M are reduced by the factor

$$1-\frac{2Q^3}{3\pi}\int_0^\infty \frac{\kappa^2\Delta(\kappa)}{1-\Delta(\kappa)}\mathrm{d}\kappa.$$

The asymptotic estimates in (23) ensure convergence of the integral and suggest the approximation

$$\frac{\Delta(\kappa)}{1-\Delta(\kappa)} \simeq \frac{1}{2\kappa^2(1+32\kappa^2/5)},$$

which yields the fractional reduction $T^{3/2}\sqrt{5}/12 \simeq 0{\cdot}186T^{3/2}$, due to the rotation.

The solutions of the two title problems discussed below are best constructed as perturbations of one or two superposed 'Sampson flows with rotation'. For this, the above velocity field and associated functions are denoted by $\mathbf{v}^*$, A^*, F^*.

2.1. Tranter's method

It is of interest to demonstrate, via the above symmetric case, that Tranter's method has a fundamentally similar solution structure. First, continuity of pressure in the hole is imposed by multiplying (19) by

$$\frac{\Gamma(n+1)}{\Gamma(n+3/2)}\left(\frac{1-r^2}{2}\right)^{1/2} r\mathcal{F}(\frac{3}{2},1;r^2),$$

where, in the notation of [14, Equation 2.1.21], $\mathcal{F}$ denotes a Jacobi polynomial, and integrating from $r=0$ to $r=1$. This yields

$$\frac{2}{\pi}\int_0^\infty k\frac{J_{2n+3/2}(k)}{k^{3/2}}A(k)\mathrm{d}k = \left(\frac{2}{\pi}\right)^{1/2}\frac{\delta_{n0}}{3}\quad (n\ge 0). \tag{29}$$

Then the no slip condition is identically satisfied by the representation

$$A(k)\frac{2}{\Pi(k/Q)}\sum_{j=1}^{3}\frac{\eta_j(k/Q)}{\sqrt{\lambda_j+1}} = \left(\frac{2\pi}{k}\right)^{1/2}\sum_{m=0}^{\infty} a_m J_{2m+3/2}(k). \tag{30}$$

Hence (29) can be rearranged as

$$\frac{a_n}{4n+3}+\sum_{m=0}^{\infty} a_m\int_0^\infty \frac{\Delta(k/Q)}{1-\Delta(k/Q)}\frac{1}{k}J_{2n+3/2}(k)J_{2m+3/2}(k)\mathrm{d}k = \frac{\delta_{n0}}{6}\ \ (n\ge 0), \tag{31}$$

after substitution of (30) and use of the identity [23, Section 6.538]

$$\int_0^\infty \frac{1}{k}J_{2n+3/2}(k)J_{2m+3/2}(k)\mathrm{d}k = \frac{\delta_{nm}}{4n+3}\quad (n,m\ge 0).$$

The dimensionless flux M is now given by

$$M=\int_0^\infty \frac{J_1(k)}{k}\left(\frac{2\pi}{k}\right)^{1/2}\sum_{m=0}^{\infty} a_m J_{2m+3/2}(k)\mathrm{d}k = \frac{2}{3}a_0,$$

by use of the identity [23, Section 6.574]

$$\int_0^\infty J_\nu(x)J_{\nu+2m+1/2}(x)\left(\frac{2}{x}\right)^{3/2}\mathrm{d}x = \frac{\Gamma(\nu)}{\Gamma(\nu+\frac{3}{2})}\delta_{m0}.$$

The system (31) reduces, in the limit $Q \to 0$, to an exact diagonal system with solution $a_0 = \frac{1}{2}$, $a_n = 0 (n \geq 1)$, thus recovering the flux $\frac{1}{3}$ in the classic Sampson flow.

Neither method is applicable when two rims are present (triple equations), as in Section 3, but Tranter's method can be used for non-symmetric flows [19]. Provided the integrals involving Bessel functions can be accurately evaluated, Tranter's method has the advantage of directly obtaining a linear system of equations.

3. Pressure-driven flow through an annular hole in a rotating wall

Suppose now that there is a pressure-driven flow is through an annular hole of radii aR, $R(a < 1)$, situated at $z = 0 (a \leq r \leq 1, -\pi < \theta \leq \pi)$. When the total velocity field is written as $\mathbf{v}^*(r,z) - \mathbf{v}(r,z)$, the optimal strategy, the perturbation flow $\mathbf{v}$ due to the partial blocking of the hole has a stream function given by (17) and associated pressure field given by (18). The imposition of no-slip and continuous pressure yield modified versions of (19) and (22), namely

$$[p]_{0-}^{0+} = \frac{2}{\pi}\int_0^\infty kJ_0(kr)A(k)\mathrm{d}k = 0 \quad (a < r < 1), \tag{32}$$

$$v_z(r,0) = \frac{1}{\pi}\int_0^\infty \frac{J_0(kr)}{\Pi(k/Q)}A(k)\sum_{j=1}^{3}\frac{\eta_j(k/Q)}{\sqrt{\lambda_j+1}}\mathrm{d}k = \begin{cases} v_z^*(r,0) & (0 \leq r < a) \\ 0 & (r > 1) \end{cases} \tag{33}$$

Equations (32) and (33) constitute a set of triple integral equations [14, Chapter 6]. Since the existence of two rims precludes the representation (26), the solution by use of Abel transforms proceeds by introducing such to satisfy (32), inverting the resultant Hankel transform and substituting for $A(k)$ in (33) to obtain integral equations for the transform functions. The pressure discontinuity is allowed, according to a local similarity solution, square-root singularities at the rims, as demonstrated above for the circular hole. Hence, let new functions $S(t)$ and $X(t)$ be defined [17, 20] by

$$[p]_{0-}^{0+} = -\frac{2}{\pi r}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{tS(t)}{(t^2-r^2)^{1/2}}\mathrm{d}t \qquad (0 \leq r < a),$$

$$[p]_{0-}^{0+} = -\frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}r}\int_1^r \frac{tX(t)}{(r^2-t^2)^{1/2}}\mathrm{d}t \qquad (r > 1). \tag{34}$$

The first of Equations (34) shows immediately that the dimensional force $4aR^2(\Delta P)F_D\hat{\mathbf{z}}$ exerted by the fluid on the disk is such that

$$F_D = \frac{1}{a}\int_0^a S(t)\mathrm{d}t, \tag{35}$$

an advantage of the Abel transforms that was exploited in earlier calculations involving disks. Thus $S(t)$ is identified as a dimensionless density function for a distribution of rings of point forces acting on the fluid in the $-\hat{\mathbf{z}}$-direction. Equations (32) and (34) enable the Hankel transform to be inverted, yielding

$$A(k) = \int_0^a S(t)\cos kt\,\mathrm{d}t - \int_1^\infty X(t)\sin kt\,\mathrm{d}t \quad (k > 0). \tag{36}$$

Equations for the functions $S(t)$, $X(t)$ are obtained by imposing (33), which by using the derivative of (20) and the additional identity [23, Section 6.554]

$$\frac{\mathrm{d}}{\mathrm{d}u}\int_u^\infty J_0(kr)\frac{r\mathrm{d}r}{(r^2-u^2)^{1/2}} = -\sin ku, \tag{37}$$

shows that no flow through the wall is achieved by setting

$$\int_0^\infty A(k)\cos ku\,\mathrm{d}k = \int_0^\infty A(k)\Delta(k/Q)\cos ku\,\mathrm{d}k + 2\pi\frac{\mathrm{d}}{\mathrm{d}u}\int_0^u \frac{r v_z^*(r,0)}{(u^2-r^2)^{1/2}}\mathrm{d}r$$

$$= \int_0^\infty A(k)\Delta(k/Q)\cos ku\,\mathrm{d}k + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}u}\int_0^1 \frac{F^*(t)}{t}\log\left(\frac{t+u}{|t-u|}\right)\mathrm{d}t \qquad (0 \le r < a),$$

$$\int_0^\infty A(k)\sin ku\,\mathrm{d}k = \int_0^\infty A(k)\Delta(k/Q)\sin ku\,\mathrm{d}k \qquad (u > 1), \tag{38}$$

where $\Delta(\kappa)$ is defined by (23) and (26) has been substituted. In the non-rotating limit, $F^*(t) = t^2$ and the forcing term reduces to $-1 - \frac{u}{2}\log\left(\frac{1+u}{1-u}\right)$, as in [20].

On substitution of (36) in Equations (38), the left-hand sides can be simplified, as in [20], to yield the pair of integral equations

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}u}\int_0^1 \frac{F^*(t)}{t}\log\left(\frac{t+u}{|t-u|}\right)\mathrm{d}t - \frac{\pi}{2}S(u) + \int_1^\infty X(t)\frac{t}{t^2-u^2}\mathrm{d}t =$$

$$\int_0^\infty\left[-\int_0^a S(t)\cos kt\mathrm{d}t + \int_1^\infty X(t)\sin kt\,\mathrm{d}t\right]\Delta(k/Q)\cos ku\,\mathrm{d}k \qquad (0 \le u < a), \tag{39}$$

$$\frac{\pi}{2}X(u) - \int_0^a S(t)\frac{u}{u^2-t^2}\mathrm{d}t =$$

$$\int_0^\infty\left[-\int_0^a S(t)\cos kt\,\mathrm{d}t + \int_1^\infty X(t)\sin kt\,\mathrm{d}t\right]\Delta(k/Q)\sin ku\,\mathrm{d}k \qquad (u > 1). \tag{40}$$

To evaluate the flux of fluid through the annular hole, consider the velocity profile in the hole $z = 0, a < r < 1$. With the definition (23), (28) and (33) yield

$$v_z^*(r,0) - v_z(r,0) = \frac{1}{2\pi}\int_r^1 \frac{F^*(t)\mathrm{d}t}{t(t^2-r^2)^{1/2}} - \frac{1}{2\pi}\int_0^\infty J_0(kr)A(k)[1-\Delta(k/Q)]\mathrm{d}k. \tag{41}$$

But the substitution of (40) allows (36) to be rewritten as

$$A(k) = \int_0^a S(t)\cos kt\,\mathrm{d}t - \frac{2}{\pi}\int_1^\infty\left[\int_0^a S(t)\frac{u}{u^2-t^2}\mathrm{d}t - \int_0^\infty A(K)\Delta(K/Q)\sin Ku\,\mathrm{d}K\right]\sin ku\,\mathrm{d}u$$

$$= A(k)\Delta(k/Q) + \frac{2}{\pi}\int_0^1\left[\int_0^a S(t)\frac{u}{u^2-t^2}\mathrm{d}t - \int_0^\infty A(K)\Delta(K/Q)\sin Ku\,\mathrm{d}K\right]\sin ku\,\mathrm{d}u,$$

$$(k > 0).$$

This effects a simplification in (41) which facilitates closed-form evaluations of the Bessel-function integrals, yielding

$$v_z^*(r,0) - v_z(r,0) = \frac{1}{2\pi}\int_r^1 \frac{F^*(t)\mathrm{d}t}{t(t^2-r^2)^{1/2}} - \frac{1}{\pi^2}\int_r^1 \frac{\mathrm{d}u}{(u^2-r^2)^{1/2}}\left[\int_0^a S(t)\frac{u}{u^2-t^2}\mathrm{d}t - \int_0^\infty A(k)\Delta(k/Q)\sin ku\,\mathrm{d}k\right], \qquad (a < r < 1).$$

Thus the dimensional flux $MR^3\Delta P/\mu$ is given by

$$\begin{aligned} M = &-\frac{2}{\pi}\int_0^a S(t)\left\{(1-a^2)^{1/2} - (a^2-t^2)^{1/2}\arcsin\left[\left(\frac{1-a^2}{1-t^2}\right)^{1/2}\right]\right\}\mathrm{d}t \\ &+\int_a^1 \frac{F^*(t)}{t}(t^2-a^2)^{1/2}\mathrm{d}t + \frac{2}{\pi}\int_0^\infty A(k)\Delta(k/Q)\int_a^1 (u^2-a^2)^{1/2}\sin ku\,\mathrm{d}u\,\mathrm{d}k, \end{aligned} \tag{42}$$

in which $A(k)$ is given in terms of S, X by (36). Comparison with [20, Equation (3.12)], shows that the effects of rotation are to modify both $F^*(t)$ and $S(t)$ and to add the last integral, which contains the effects of both the rotation and the partially blocked hole.

The $O(Q)$ perturbation can be calculated from the approximate forms of Equations (39) and (40):

$$1 - \frac{u}{2}\log\left(\frac{1+u}{1-u}\right) - \frac{\pi}{2}S(u) + \int_1^\infty X(t)\frac{t}{t^2-u^2}\mathrm{d}t =$$

$$-Q\int_0^a S(t)\mathrm{d}t\int_0^\infty \Delta(\kappa)\mathrm{d}\kappa \quad (0 \le u < a),$$

$$\frac{\pi}{2}X(u) - \int_0^a S(t)\frac{u}{u^2-t^2}\mathrm{d}t = 0, \quad (r > 1),$$

from which $X(t)$ can be eliminated, as in [20], to yield an integral equation for $S(u)$. On setting $\Delta(\kappa) \simeq (1 + 64\kappa^2/5)^{-1}$, in accordance with (23), and writing

$$S(u) \simeq S_0(u) + Qa\frac{5^{1/4}}{4}C_0 S_1(u),$$

where $S_0(u)$ is the zero-order solution given by [20] and $C_0(a)$ is its mean value, it is found that $S_1(u)$ satisfies

$$S_1(u) - \frac{2}{\pi^2}\int_0^a S_1(v)\left[u\log\left(\frac{1+u}{1-u}\right) - v\log\left(\frac{1+v}{1-v}\right)\right]\frac{\mathrm{d}v}{u^2-v^2} = 1 \quad (0 \le u < a). \tag{43}$$

According to (35) and (42), the dimensionless force on the disk and flux through the annular hole are given, in this approximation, by

$$F_D = C_0 + \left[Qa\frac{5^{1/4}}{4}C_0\right]C_1, \qquad M = M_0 - \left[Qa\frac{5^{1/4}}{4}C_0\right]M_1,$$

where, by numerical solution of (43), C_1 and M_1 take values 1, 1·14, 1·36, 1·96 and 0, 0·14, 0·17, 0·068 at $a = 0, 0·3, 0·6, 0·9$, respectively. The previously computed C_0, M_0 are such that

C_0 increases almost linearly from $2/\pi$ to $\pi/4$ and $M_0(1-a^2)^{-3/2}$ decreases almost linearly from 1/3 to 0 as a increases from 0 to 1. $M_1 \to 0$ less rapidly than M_0 as $a \to 1$.

4. Pressure-driven flow through two planes

Suppose that there is a pressure drop ΔP across each of two thin rigid planes at $z = \pm H$ (dimensional separation $2HR$), with concentric holes of unit dimensionless radius centered on the z-axis. Then the flow relative to the rotating axes is $\mathbf{v}^*(r, z-H)+\mathbf{v}^*(r, z+H)-\mathbf{v}(r, z)$, with $\mathbf{v}(r, z)$ described by the stream function $\psi(r, z)$, even in z, given by

$$\psi = \frac{r}{\pi}\int_0^\infty \frac{J_1(kr)}{k\Pi(k/Q)}\sum_{j=1}^{3}\eta_j(k/Q)\mathrm{e}^{-k(|z|-H)\sqrt{\lambda_j+1}}\left[\frac{A(k)}{\sqrt{\lambda_j+1}} + B(k) - \frac{Q^2C(k)}{k^2\lambda_j}\right]\mathrm{d}k,$$
$$(|z| < H), \tag{44}$$

$$\begin{aligned}\psi = \frac{r}{\pi}\int_0^\infty \frac{J_1(kr)}{k\Pi(k/Q)}\sum_{j=1}^{3}\eta_j(k/Q)\cosh(kz\sqrt{\lambda_j+1})\Bigg[&\frac{A_1(k)}{\sqrt{\lambda_j+1}\sinh(kH\sqrt{\lambda_j+1})}\\ &-\left\{B_1(k) - \frac{Q^2C_1(k)}{k^2\lambda_j}\right\}\frac{1}{\cosh(kH\sqrt{\lambda_j+1})}\Bigg]\mathrm{d}k, \qquad (|z| < H),\end{aligned} \tag{45}$$

in which, compared to (17), functions that generate non-zero values of v_r, v_θ at the walls must now be included because neither lies in a plane of symmetry of the flow. Then it is readily deduced from (3), (5) and (6) that

$$v_z, \quad L_{-1}\psi, \quad v_r, \quad p, \quad v_\theta,$$

differ from the ψ expressions in (44) and (45) by having the factor $rJ(kr)/k$ replaced by

$$J_0(kr), \quad rkJ_1(kr)\lambda_j, \quad \mathrm{sgn}z\,J_1(kr)\sqrt{\lambda_1+1}, \quad -\mathrm{sgn}z\,kJ_0(kr)\frac{\lambda_j}{\sqrt{\lambda_1+1}},$$
$$-\mathrm{sgn}z\frac{k^2}{Q^2}J_1(kr)\frac{\lambda_j^2}{\sqrt{\lambda_j+1}} = \mathrm{sgn}z\frac{Q^2}{k^2}J_1(kr)\frac{\sqrt{\lambda_j+1}}{\lambda_j}, \qquad (|z| > H),$$

respectively, or the factor $r[J_1(kr)/k]\cosh(kz\sqrt{\lambda_j+1})$ by

$$\left[J_0(kr), \quad rkJ_1(kr)\lambda_j\right]\cosh(kz\sqrt{\lambda_j+1}),$$
$$\left[-J_1(kr)\sqrt{\lambda_j+1},\ kJ_0(kr)\frac{\lambda_j}{\sqrt{\lambda_j+1}},\ -\frac{Q^2}{k^2}J_1(kr)\frac{\sqrt{\lambda_j+1}}{\lambda_j}\right]\sinh(kz\sqrt{\lambda_j+1}),$$
$$(|z| < H),$$

respectively. Evidently, the appearance of the hyperbolic functions complicates the subsequent algebra, whose presentation is abbreviated by defining the following sums:

$$\Sigma_0(\kappa) = \frac{2}{\Pi(\kappa)}\sum_{j=1}^{3}\frac{\eta_j(\kappa)}{\sqrt{\lambda_j+1}}, \quad \Sigma_{0H}(\kappa, QH) = \frac{2}{\Pi(\kappa)}\sum_{j=1}^{3}\frac{\eta_j(\kappa)}{\sqrt{\lambda_j+1}}e^{-2\kappa QH\sqrt{\lambda_j+1}},$$

$$\Sigma_1(\kappa) = \frac{2}{\Pi(\kappa)} \sum_{j=1}^{3} \frac{\eta_j(\kappa)\lambda_j}{\sqrt{\lambda_j + 1}}, \quad \Sigma_2(\kappa) = \frac{2}{\Pi(\kappa)} \sum_{j=1}^{3} \frac{\eta_j(\kappa)}{\lambda_j}\sqrt{\lambda_j + 1},$$

$$\begin{bmatrix} \sigma_{0c}(\kappa, QH) & \sigma_{1c}(\kappa, QH) & \sigma_{2c}(\kappa, QH) \\ \sigma_{0t}(\kappa, QH) & \sigma_{1t}(\kappa, QH) & \sigma_{2t}(\kappa, QH) \end{bmatrix} =$$

$$\frac{2}{\Pi(\kappa)} \sum_{j=1}^{3} \eta_j(\kappa) \left[\frac{1}{\sqrt{\lambda_j + 1}}, \ \frac{\lambda_j}{\sqrt{\lambda_j + 1}}, \ \frac{\sqrt{\lambda_j + 1}}{\lambda_j} \right] \begin{bmatrix} \coth(\kappa QH\sqrt{\lambda_j + 1}) - 1 \\ 1 - \tanh(\kappa QH\sqrt{\lambda_j + 1}) \end{bmatrix}.$$

Equations relating the functions A_1, B_1, C_1 and A, B, C are obtained by imposing continuity of $\mathbf{v}$ at $z = H$. Thus the velocity components derived from (44), (45) show that

$$A_1(k)[\Sigma_0(k/Q) + \sigma_{0c}(k/Q, QH)] = A(k)\Sigma_0(k/Q),$$

$$B_1(k)[\Sigma_0(k/Q) + \Sigma_1(k/Q) - \sigma_{0t}(k/Q, QH) - \sigma_{1t}(k/Q, QH)] -$$

$$\frac{Q^2}{k^2} C_1(k)[\Sigma_2(k/Q) - \sigma_{2t}(k/Q, QH)] = B(k)[\Sigma_0(k/Q) + \Sigma_1(k/Q)] - \frac{Q^2}{k^2} C(k)\Sigma_2(k/Q),$$

$$\frac{Q^2}{k^2} B_1(k)[\Sigma_2(k/Q) - \sigma_{2t}(k/Q, QH)] + C_1(k)[\Sigma_1(k/Q) - \sigma_{1t}(k/Q, QH)]$$

$$= \frac{Q^2}{k^2} B(k)\Sigma_2(k/Q) + C(k)\Sigma_1(k/Q).$$

It is also convenient to write

$$\delta A = A - A_1, \quad \delta B = B - B_1, \quad \delta C = C - C_,$$

and introduce the matrices

$$M(\kappa) = \begin{bmatrix} \Sigma_0(\kappa) + \Sigma_1(\kappa) & -\frac{1}{\kappa^2}\Sigma_2(\kappa) \\ \frac{1}{\kappa^2}\Sigma_2(\kappa, QH) & \Sigma_1(\kappa) \end{bmatrix},$$

$$M_t(\kappa, QH) = \begin{bmatrix} \sigma_{0t}(\kappa, QH) + \sigma_{1t}(\kappa, QH) & -\frac{1}{\kappa^2}\sigma_{2t}(\kappa, QH) \\ \frac{1}{\kappa^2}\sigma_{2t}(\kappa, QH) & \sigma_{1t}(\kappa, QH) \end{bmatrix}.$$

Then

$$\delta A(k) = \frac{\sigma_{0c}(k/Q, QH)}{\Sigma_0(k/Q)} A_1(k), \quad \begin{bmatrix} \delta B(k) \\ \delta C(k) \end{bmatrix} = -M^{-1}(k/Q) M_t(k/Q, QH) \begin{bmatrix} B_1(k) \\ C_1(k) \end{bmatrix}. \quad (46)$$

The pressure, vorticity and tangential stress discontinuities at $z = H$, derived from (44), (45), reduce, because (13), (15) and (16), to

$$[p]_{H-}^{H+} = \frac{1}{\pi} \int_0^\infty k J_0(kr)\{A(k) + A_1(k) - \frac{1}{2}[\delta B(k)\Sigma_1(k/Q) + B_1(k)\sigma_{1t}(k/Q, QH)]$$
$$+ \frac{Q^2}{2k^2}[\delta C(k)\Sigma_1(k/Q) + C_1(k)\sigma_{1t}(k/Q, QH)]\}\mathrm{d}k, \quad (47)$$

$$[-r^{-1}L_{-1}\psi]_{H-}^{H+} = \frac{1}{\pi}\int_0^\infty kJ_1(kr)\{B(k) + B_1(k) - \frac{1}{2}[\delta A(k)\Sigma_1(k/Q) - A_1(k)\sigma_{1c}(k/Q, QH)]\}\mathrm{d}k$$

$$\left[\frac{\partial v_\theta}{\partial z}\right]_{H-}^{H+} = \frac{1}{\pi}\int_0^\infty kJ_1(kr)\{C(k) + C_1(k) - \frac{Q^2}{2k^2}[\delta A(k)\Sigma_2(k/Q) - A_1(k)\sigma_{2c}(k/Q, QH)]\}\mathrm{d}k. \tag{48}$$

The mixed-boundary-value problem, which determines the functions $A_1(k), B_1(k), C_1(k)$, is formulated by setting the three velocity components to zero in $r > 1$ and eliminating the stress discontinuities in $r < 1$. The pressure discontinuity and normal velocity can be handled by the simpler method employed above for the circular hole but the tangential velocities and the vorticity and tangential stress discontinuities require the use of Abel transforms. These latter discontinuities are confined to the wall ($r > 1$), with square-root singularities at the rim, by defining [15] new functions $W(t)$ and $Y(t)$ by

$$\begin{bmatrix} -r^{-1}L_{-1}\psi \\ \partial v_\theta/\partial z \end{bmatrix}_{H-}^{H+} = -\frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}r}\int_1^r \begin{bmatrix} W(t) \\ Y(t) \end{bmatrix}\frac{\mathrm{d}t}{(r^2-t^2)^{1/2}}, \quad (r > 1). \tag{49}$$

Then the inversion of the Hankel transforms in Equations (48), followed by use of the additional identity [23, Section 6.552]

$$\frac{\mathrm{d}}{\mathrm{d}u}\int_0^\infty J_1(kr)\frac{u\mathrm{d}r}{(r^2-u^2)^{1/2}} = \cos ku, \tag{50}$$

yields

$$\begin{bmatrix} \delta B(k) + 2B_1(k) - \frac{1}{2}[\delta A(k)\Sigma_1(k/Q) - A_1(k)\sigma_{1c}(k/Q, QH)] \\ \delta C(k) + 2C_1(k) - \frac{Q^2}{2k^2}[\delta A(k)\Sigma_2(k/Q) - A_1(k)\sigma_{2c}(k/Q, QH)] \end{bmatrix} = 2\int_1^\infty \begin{bmatrix} W(t) \\ Y(t) \end{bmatrix}\cos kt\,\mathrm{d}t \quad (k > 0).$$

Thus

$$\begin{bmatrix} B_1(k) \\ C_1(k) \end{bmatrix} = [2M(k/Q) - M_t(k/Q)]^{-1}M(k/Q)\left\{2\int_1^\infty \begin{bmatrix} W(t) \\ Y(t) \end{bmatrix}\cos kt\,\mathrm{d}t + \frac{1}{2}\begin{bmatrix} \frac{\Sigma_1(k/Q)}{\Sigma_0(k/Q)}\sigma_{0c}(k/Q, QH) - \sigma_{1c}(k/Q, QH) \\ \frac{Q^2}{2k^2}\left[\frac{\Sigma_2(k/Q)}{\Sigma_0(k/Q)}\sigma_{0c}(k/Q, QH) - \sigma_{2c}(k/Q, QH)\right] \end{bmatrix} A_1(k)\right\} \tag{51}$$

Mimicking Section 2, the no-slip condition,

$$v_z(r, H) = v_z^*(r, H) = \frac{1}{2}\int_0^\infty J_0(kr)A^*(k)\Sigma_{0H}(k/Q, QH)\,\mathrm{d}k \quad (r > 1),$$

is satisfied by setting, by reference to (22), (26),

$$A(k)\Sigma_0(k/Q) = A^*(k)\Sigma_{0H}(k/Q, QH) - \int_0^1 t^{-1}F(t)\sin kt\,\mathrm{d}t \qquad (k > 0), \tag{52}$$

whence the total dimensionless flux through each hole is given by

$$M = \int_0^1 [F^*(k) + F(t)]\mathrm{d}t. \tag{53}$$

Equations for the functions $F(t)(0 \le t < 1)$ and $W(t), Y(t)(t > 1)$ are obtained by imposing pressure continuity at $0 \le r < 1$ and the remaining no-slip conditions at the wall $r > 1$. The equation corresponding to (27) is found, from (47), to be

$$\int_0^\infty \sin ku \left\{ \left[2 + \frac{\sigma_{0c}(k/Q, QH)}{\Sigma_0(k/Q)} \right] A_1(k) + \frac{1}{2} \left\langle \left[\Sigma_1(k/Q), -\frac{Q^2}{k^2}\Sigma_0(k/Q) \right] M(k/Q)^{-1} M_t(k/Q) \right.\right.$$
$$\left.\left. - [\sigma_{1t}(k/Q, QH), -\frac{Q^2}{k^2}\sigma_{0t}(k/Q, QH)] \right\rangle \right\} \begin{bmatrix} B_1(k) \\ C_1(k) \end{bmatrix} \mathrm{d}k = 0 \quad (0 \le u < 1), \tag{54}$$

after substitution of (46). The conditions

$$v_r(r, H) = v_r^*(r, H), \qquad v_\theta(r, H) = v_\theta^*(r, H),$$

with the velocities derived from (44) and (17), reduce, after application of (50) and further substitution of (46), to

$$\int_0^\infty \cos ku[M(k/Q) - M_t(k/Q)] \begin{bmatrix} B_1(k) \\ C_1(k) \end{bmatrix}$$
$$= \int_0^\infty \frac{\cos ku}{\Pi(\kappa)} \sum_{j=1}^3 \frac{\eta_j(k)}{\sqrt{\lambda_j + 1}} \mathrm{e}^{-2\kappa QH)\sqrt{\lambda_j+1}} \begin{bmatrix} 1 \\ \frac{Q^2}{k^2\lambda_j} \end{bmatrix} A^*(k)\, \mathrm{d}k \quad (u > 1). \tag{55}$$

On substitution of (51), (52) in (54), (55), a set of integral equations for $F(t)(0 \le t < 1)$ and $W(t), Y(t)(t > 1)$ is obtained. Only $F(t)$ is required for determining the flux, according to (53).

5. Conclusions

Two problems in rotating, viscous fluids have each been reduced, by use of Abel transforms, to the solution of sets of integral equations. Earlier, the pressure-driven flow through a hole in a rotating plane was determined by reduction to an integral equation or a set of linear equations. It was shown that the introduction of rotation initially reduces the flux from its value in the classic Sampson flow. Tranter's method can be applied to the two-plane problem but not to the annular hole because it can cope with only one edge. When applicable, it has the advantage that a linear system of equations, albeit with coefficients that are numerically evaluated integrals, presents less computational difficulty than sets of integral equations that not only have numerically evaluated kernels but also must be discretized.

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